

ON THE EQUIVALENCE OF FSF AND WEAKLY LASKERIAN CLASSES

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ABSTRACT. It is proved that, over a Noetherian ring R , the class of weakly Laskerian and FSF modules are the same classes. By using this characterization we proved that the property of being weakly Laskerian descends by finite integral extensions of local ring homomorphisms and ascends by tensoring under the completion.

Introduction

Throughout this paper, R will denote a commutative ring with identity element 1_R and all modules are considered unitary. It is well known that if R is a Noetherian ring and M a finitely generated R -module, then for all proper submodule N of M the set of associated prime ideals of M/N , abbreviated as $\text{Ass}_R(M/N)$ or $\text{Ass}(M/N)$ if there is no confusion about the underlying ring R , is a finite set. In fact, the class of Laskerian modules has the property that the set of associated prime ideals of any quotient of a Laskerian module is a finite set and it is not hard to find examples of Laskerian modules which is not finitely generated. Recall that an R -module is called *Laskerian* if every proper submodule is an intersection of a finite number of primary submodules. The importance of the finiteness of $\text{Ass}(M/N)$, for a submodule N of an R -module M , comes back to the fact that the finiteness of $\text{Ass}(M/N)$ guarantees the existence of regular elements on M/N , which is of the most importance in the study of cohomological properties of M/N . Thinking of these facts and toward a general study of Local cohomology modules in [3], Divaani-Aazar and Mafi introduced the class of weakly Laskerian modules and proved some interesting properties of their local cohomology modules which were known just for finite modules. Recall that an R -module M is called *weakly Laskerian* if $\text{Ass}(M/N)$ is a finite set for each proper submodule N of M . It is easy to see that the class of Artinian, Laskerian and Minimax modules are contained in the class of weakly Laskerian modules. Recently, Hung Quy [5], introduced the class of FSF modules, modules containing some finitely generated submodules such that the support of the quotient module is finite, and proved the following result.

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Theorem 0.1. *Assume that \mathfrak{a} is an ideal of the Noetherian ring R , and let M be a finitely generated R -module. Let $t \in \mathbb{N}_0$ be such that either $H_{\mathfrak{a}}^i(M)$ is finite or $\text{Supp}(H_{\mathfrak{a}}^i(M))$ is finite for all $i < t$. Then $\text{Ass}(H_{\mathfrak{a}}^t(M))$ is finite.*

See also [2, Proposition 2.1]. The main purpose of this paper is to prove that over a Noetherian ring R , for an R -module the property of being weakly Laskerian or FSF are the same. By using this characterization we prove that the property of being weakly Laskerian ascends by tensoring under the completion and descends by finite integral extensions of local ring homomorphisms. More precisely, it is proved that:

Theorem 0.2. *Let R be a Noetherian ring and M a nonzero R -module. The following statements are equivalent:*

- (1) *M is a weakly Laskerian module;*
- (2) *M is an FSF module.*

Theorem 0.3. *Let (R, \mathfrak{m}) be a Noetherian local ring and M a weakly Laskerian R -module. Then $M \otimes_R R^*$ is a weakly Laskerian R^* -module, where R^* denotes the \mathfrak{m} -adic completion of R .*

Theorem 0.4. *Let $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a Noetherian local ring homomorphism in which S is a finite R -module. Then every weakly Laskerian S -module is a weakly Laskerian R -module.*

1. Preliminaries

In this section we bring some technical results, which will be used later.

The following Lemma is a well known fact [1, Proposition 2.1.4], but the proof which is given here is elementary and included for completeness.

Definition 1.1. *Let R be ring and I an ideal of R . For an R -module M the I -torsion submodule of M is denoted by $\Gamma_I(M)$ and defines as*

$$\Gamma_I(M) = \{x \in M \mid \exists n \in \mathbb{N} \text{ such that } I^n x = 0\}.$$

Lemma 1.2. *Let R be a Noetherian ring, and I an ideal of R . If E is an injective R -module, then so is $\Gamma_I(E)$ an injective R -module.*

Proof. Let E_1 be the injective envelope of $\Gamma_I(E)$. It is enough to prove that $E_1 = \Gamma_I(E)$. By the way of contradiction, assume that E_1 properly contains $\Gamma_I(E)$ and let

$$\mathcal{S} = \{(0 :_R e) \mid e \in E_1 \setminus \Gamma_I(E)\}.$$

R being a Noetherian ring implies that \mathcal{S} has, at least, a maximal element, $\mathfrak{p} = (0 :_R x)$ say. We claim that \mathfrak{p} is a prime ideal. Assume the contrary.

Then, for some $a, b \in R$, we have $ab \in \mathfrak{p}$, while neither a nor b does not belong to \mathfrak{p} . Since $\mathfrak{p} = 0 :_R x$ is a proper subset of $(0 :_R bx)$ and \mathfrak{p} is a maximal element of \mathcal{S} , then $bx \in \Gamma_I(E)$. Therefore, there exists a natural integer t such that $I^t bx = 0$. If $I^t = (c_1, \dots, c_n)$, then $bc_k x = 0$ for all $k = 1, \dots, n$. Again, as discussed above, by maximality of \mathfrak{p} for all $k = 1, \dots, n$ there exists an integer $t_k \in \mathbb{N}$ such that $I^{t_k} c_k x = 0$. Set $s := \max_{1 \leq i \leq n} t_i$. It is easy to see that $I^{s+t} x = 0$, which contradicts the assumption that x does not belong to $\Gamma_I(E)$. Therefore, $\mathfrak{p} = 0 :_R x$ is a prime ideal of R . Since E_1 is an essential extension of $\Gamma_I(E)$, then there exists $r \in R$ such that $0 \neq rx \in \Gamma_I(E)$. As r does not belong to \mathfrak{p} and \mathfrak{p} is a prime ideal, then $I \subseteq \mathfrak{p}$ which contradicts the fact that

$$x \notin \Gamma_I(E) = \Gamma_I(E_1) \subseteq E_1 \subseteq E,$$

where the last containment is true by injectivity of E . □

In what follows the next theorem plays an important role. Before stating it, to simplify our expressions, we give a definition.

Definition 1.3. Let R be a ring and k a nonnegative integer. We define

$$A(R, k) := \{\mathfrak{p} \in \text{Spec } R \mid \dim(R/\mathfrak{p}) = k\},$$

where $\text{Spec } R$ denotes the prime spectrum of R .

Theorem 1.4. Assume that R be a Noetherian ring, k a nonnegative integer and let M be an R -module such that the intersection $\text{Supp}(M) \cap A(R, k)$ is an infinite set. If $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals of R such that

$$\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subseteq \text{Ass}(M) \cap A(R, k),$$

then there exists a prime ideal, say \mathfrak{p}_{n+1} , and a nonzero elements of M , x say, satisfying the following conditions:

- (1) $\mathfrak{p}_{n+1} \in \text{Supp}(M) \cap A(R, k) \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$,
- (2) $(0 :_R x) \subseteq \mathfrak{p}_{n+1}$,
- (3) $Rx \cap \Gamma_{\mathfrak{p}_1 \dots \mathfrak{p}_n}(M) = 0$.

Proof. Since the intersection $\text{Supp}(M) \cap A(R, k)$ has infinitely many elements, then it is possible to find a prime ideal $\mathfrak{p}_{n+1} \in A(R, k) \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ such that $0 :_R y \subseteq \mathfrak{p}_{n+1}$ for some $y \in M$. By Lemma [1], $y = a + b$, in which $a, b \in E_R(M)$,

$$Rb \cap \Gamma_{\mathfrak{p}_1 \dots \mathfrak{p}_n}(M) = 0 \text{ and } (\mathfrak{p}_1 \dots \mathfrak{p}_n)^t a = 0 \quad (\dagger)$$

for some natural integer t . Therefore,

$$(\mathfrak{p}_1 \dots \mathfrak{p}_n)^t y = (\mathfrak{p}_1 \dots \mathfrak{p}_n)^t b.$$

We claim that $0 :_R (\mathfrak{p}_1 \dots \mathfrak{p}_n)^t b \subseteq \mathfrak{p}_{n+1}$. Assume the contrary, and let $s \in R \setminus \mathfrak{p}_{n+1}$ such that $s(\mathfrak{p}_1 \dots \mathfrak{p}_n)^t y = s(\mathfrak{p}_1 \dots \mathfrak{p}_n)^t b = 0$. Since \mathfrak{p}_{n+1} is a prime

ideal then $\mathfrak{p}_j \subseteq \mathfrak{p}_{n+1}$, for some $j = 1, \dots, n$. In view of the fact that

$$\dim R/\mathfrak{p}_j = k = \dim R/\mathfrak{p}_{n+1},$$

we conclude that $\mathfrak{p}_j = \mathfrak{p}_{n+1}$ which is the desired contradiction. Therefore,

$$0 :_R (\mathfrak{p}_1 \dots \mathfrak{p}_n)^t y \subseteq \mathfrak{p}_{n+1}.$$

Since $(\mathfrak{p}_1 \dots \mathfrak{p}_n)^t b$ is a finitely generated R -module then $0 :_R x \subseteq \mathfrak{p}_{n+1}$, for some $x \in (\mathfrak{p}_1 \dots \mathfrak{p}_n)^t b$. On the other hand,

$$Rx \cap \Gamma_{\mathfrak{p}_1 \dots \mathfrak{p}_n}(M) \subseteq (\mathfrak{p}_1 \dots \mathfrak{p}_n)^t b \cap \Gamma_{\mathfrak{p}_1 \dots \mathfrak{p}_n}(M) \subseteq Rb \cap \Gamma_{\mathfrak{p}_1 \dots \mathfrak{p}_n}(M) = 0,$$

where the last equality is true by (\dagger) , and the proof is completed. \square

Theorem 1.5. *Assume that R is a Noetherian ring, k a nonnegative integer and let M be an R -module such that for all finitely generated submodule N of M the intersection $\text{Supp}(M/N) \cap A(R, k)$ is not a finite set. Then there exists a submodule L of M such that the intersection $\text{Ass}(M/L) \cap A(R, k)$ is an infinite set.*

Proof. First of all, we want to show that for each natural integer j there exists a chain $T_1 \subseteq T_2 \subseteq \dots \subseteq T_j$, of finitely generated submodules of M , prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_j \in A(R, k)$ and elements x_1, \dots, x_j such that $\mathfrak{p}_i = T_j :_R x_i + T_j$, for each $i = 1, \dots, j$. We use induction on j . Let \mathfrak{p}_1 be an arbitrary element of $\text{Supp}(M) \cap A(R, k)$. There is an element $x_1 \in M$ such that $0 :_R x_1 \subseteq \mathfrak{p}_1$. Set $T_1 := \mathfrak{p}_1 x_1$. Obviously, $\mathfrak{p}_1 \in \text{Ass}(M/T_1)$ and, for $j = 1$, we are done. Now, we are going to construct T_{j+1} . Since, by induction hypothesis, T_j is finitely generated and the intersection $\text{Supp}(M/T_j) \cap A(R, k)$ is an infinite set then by Theorem [1.4], there exists a prime ideal

$$\mathfrak{p}_{j+1} \in \text{Supp}(M/T_j) \cap A(R, k) \setminus \{\mathfrak{p}_1, \dots, \mathfrak{p}_j\},$$

and a nonzero element $x_{j+1} + T_j \in M/T_j$ such that

$$(T_j :_R x_{j+1} + T_j) \subseteq \mathfrak{p}_{j+1} \text{ and } (Rx_{j+1} + T_j)/T_j \cap \Gamma_{\mathfrak{p}_1 \dots \mathfrak{p}_j}(M/T_j) = 0.$$

Set $T_{j+1} := \mathfrak{p}_{j+1} x_{j+1} + T_j$. First, we prove that $(T_{j+1} :_R x_i + T_{j+1}) = \mathfrak{p}_i$, for $i = 1, \dots, j+1$. In the case that $i = j+1$, one just need to investigate that $(T_{j+1} :_R x_{j+1} + T_{j+1})$ is a subset of \mathfrak{p}_{j+1} . Assume the contrary, and let $s \in R \setminus \mathfrak{p}_{j+1}$ be such that $sx_{j+1} \in T_{j+1} = \mathfrak{p}_{j+1} x_{j+1} + T_j$. Then

$$s(Rx_{j+1} + T_j)/T_j \subseteq (\mathfrak{p}_{j+1} x_{j+1} + T_j)/T_j = \mathfrak{p}_{j+1}(Rx_{j+1} + T_j)/T_j.$$

This implies that

$$s \in \sqrt{\mathfrak{p}_{j+1} + (T_j :_R Rx_{j+1})} = \mathfrak{p}_{j+1},$$

which is a contradiction. Therefore, $(T_{j+1} :_R x_{j+1} + T_{j+1}) = \mathfrak{p}_{j+1}$. On the other hand, by induction hypothesis, for all $i = 1, \dots, j$ and $s \in R \setminus \mathfrak{p}_i$ we have $0 \neq sx_i + T_j \in \Gamma_{\mathfrak{p}_1 \dots \mathfrak{p}_j}(M/T_j)$. Since

$$(Rx_{j+1} + T_j)/T_j \cap \Gamma_{\mathfrak{p}_1 \dots \mathfrak{p}_j}(M/T_j) = 0,$$

then

$$(Rx_{j+1} + T_j) \cap (Rs x_i + T_j) = T_j.$$

This means that sx_i does not belong to $Rx_{j+1} + T_j$, which implies that sx_i does not belong to T_{j+1} . Therefore,

$$(T_{j+1} :_R x_i + T_{j+1}) = \mathfrak{p}_i, i = 1, 2, \dots, j.$$

Set $L := \cup_{j=1}^{\infty} T_j$. We claim that

$$\bigcup_{j=1}^{\infty} \{\mathfrak{p}_j\} \subseteq \text{Ass}(M/L) \cap A(R, k).$$

It is enough to prove that $(L :_R x_j + L) \subseteq \mathfrak{p}_j$. By the way of contradiction, assume that there exists an element $s \in R \setminus \mathfrak{p}_j$ such that $sx_j \in L$. Let l be a natural integer such that $sx_j \in T_l$. But, in this way $sx_j \in T_{j+l}$, which contradicts the fact that

$$(T_{j+l} :_R x_j + T_{j+l}) = \mathfrak{p}_j.$$

□

Corollary 1.6. *Let R be a Noetherian ring and M a weakly Laskerian R -module. Then for each nonnegative integer k , M has a finitely generated submodule, N_k say, such that the intersection $\text{Supp}(M/N_k) \cap A(R, k)$ is finite.*

Proof. By Theorem [1.4], and the definition of a weakly Laskerian module every thing is evident. □

2. Main Result

This section contains the main result of this paper asserting that over a Noetherian ring the class of FSF and weakly Laskerian modules are the same classes.

Definition 2.1. (See [3], Definition 2.1) *An R -module M is said to be weakly Laskerian, if $\text{Ass}(M/N)$ is finite, for each proper submodule N of M .*

Definition 2.2. (See [5], Definition 2.1) *An R -module M is said to be FSF, if there exists a **F**initely generated submodule N of M , such that the **S**upport of the quotient module M/N is a **F**inite set.*

Definition 2.3. *An R -module M is said to be Minimax, if there exists a finitely generated submodule N of M , such that M/N is an Artinian.*

The class of minimax modules was introduced by Zöchinger [9] and it was proved by Zink [8] and Enochs [4], that over a complete local ring a module is minimax if and only if it is Matlis reflexive.

Remark 2.4. Comparing the property of being Artinian, Laskerian, weakly Laskerian or Minimax module, over a Noetherian ring, it is not hard to see that

$$\text{Artinian} \Rightarrow \text{Laskerian} \Rightarrow \text{weakly Laskerian},$$

and

$$\text{Artinian} \Rightarrow \text{Minimax} \Rightarrow \text{weakly Laskerian}.$$

One can find examples of FSF or weakly Laskerian modules which is not Artinian, Laskerian or Minimax. However, the next theorem shows that, over a Noetherian ring, the property of being FSF or weakly Laskerian module are the same properties.

Theorem 2.5. *Let R be a Noetherian ring and M a nonzero R -module. The following statements are equivalent:*

- (1) *M is a weakly Laskerian module;*
- (2) *M is an FSF module.*

Proof. (1) \Rightarrow (2) By Corollary [1.6], there exists finitely generated submodules N_0 and N_1 such that the intersection

$$\text{Supp}(M/N_0) \cap A(R, 0) \quad \text{and} \quad \text{Supp}(M/N_1) \cap A(R, 1),$$

are finite sets. Put $N := N_0 + N_1$. It is enough to prove that

$$\text{Supp}(M/N) \subseteq (\text{Supp}(M/N_0) \cap A(R, 0)) \cup (\text{Supp}(M/N_1) \cap A(R, 1)).$$

Suppose the contrary and let \mathfrak{p} be a prime ideal maximal with respect to the property

$$\mathfrak{p} \in \text{Supp}(M/N) \setminus [(\text{Supp}(M/N_0) \cap A(R, 0)) \cup (\text{Supp}(M/N_1) \cap A(R, 1))].$$

In view of

$$\text{Supp}(M/N) \subseteq \text{Supp}(M/N_0) \cup \text{Supp}(M/N_1),$$

we conclude that $\dim(R/\mathfrak{p}) > 1$ and so $V(\mathfrak{p})$ is not finite. For each prime ideal \mathfrak{Q} which properly contains \mathfrak{p} , the maximal property of \mathfrak{p} , implies that

$$\mathfrak{Q} \in [(\text{Supp}(M/N_0) \cap A(R, 0)) \cup (\text{Supp}(M/N_1) \cap A(R, 1))],$$

which contradicts the fact that the union

$$(\text{Supp}(M/N_0) \cap A(R, 0)) \cup (\text{Supp}(M/N_1) \cap A(R, 1)),$$

is a finite set.

(2) \Rightarrow (1) There exists a finitely generated submodule N of M such that $\text{Supp}(M/N)$ is a finite set. Let L be an arbitrary submodule of M . From the short exact sequence

$$0 \rightarrow N/N \cap L \rightarrow M/L \rightarrow M/L + N \rightarrow 0,$$

we conclude that

$$\text{Ass}(M/L) \subseteq \text{Ass}(N/N \cap L) \cup \text{Ass}(M/N + L).$$

This means that $\text{Ass}(M/L)$ is a finite set and we are done. \square

Corollary 2.6. *Let (R, \mathfrak{m}) be a Noetherian local ring and M a weakly Laskerian R -module. Then $M \otimes_R R^*$ is a weakly Laskerian R^* -module, where R^* denotes the \mathfrak{m} -adic completion of R .*

Proof. By Theorem [2.5], there exists a finitely generated submodule N of M such that $\text{Supp}_R(M/N)$ is finite and so $\dim_R(M/N) \leq 1$. Set $J := \bigcap_{\mathfrak{p} \in \text{Supp}_R(M/N)} \mathfrak{p}$. Then $\dim(R/J) \leq 1$ and $\text{Supp}_R(M/N) = V(J)$. It is easy to see that,

$$\text{Supp}_{R^*}(M/N \otimes_R R^*) \subseteq V(JR^*).$$

Since $\dim R^*/JR^* \leq 1$ and R^* is a local ring then $V(JR^*)$ and, consequently, $\text{Supp}_{R^*}(M/N \otimes_R R^*)$ is a finite set. Now, by Theorem [2.5], we conclude that $M \otimes_R R^*$ is a weakly Laskerian R^* -module. \square

The following theorem shows that the property of being weakly Laskerian ascends under the local finite integral ring extensions.

Corollary 2.7. *Let $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a Noetherian local ring homomorphism in which S is a finite R -module. Then every weakly Laskerian S -module is a weakly Laskerian R -module.*

Proof. Let M be a weakly Laskerian S -module. By Theorem [2.5], there exists a finitely generated submodule of M , say N , such that $\text{Supp}_S(M/N)$ is a finite set. Set $J := \bigcap_{\mathfrak{p} \in \text{Supp}_S(M/N)} \mathfrak{p}$. Then $\text{Supp}_S(M/N) = V(J)$ and $\dim_S(M/N) = \dim(S/J) \leq 1$. Set $I = J \cap R$. Since S/J is an integral extension of R/I then, by lying over theorem, one can conclude that $\text{Supp}_R(M/N) \subseteq V(I)$. As $\dim(R/I) \leq 1$ and R is a local ring then $V(I)$ and so $\text{Supp}_R(M/N)$ is a finite set. Now, the result is evident by Theorem [2.5]. \square

REFERENCES

1. M. P. Brodmann and R. Y. Sharp: Local cohomology-An algebraic introduction with geometric applications, Cambridge. Univ. Press, 1998.
2. M. P. Brodmann and A. Lashgari Faghani, A finiteness result for associated primes of local cohomology modules. Proc. Amer. Math. Soc. 128 (2000), 2851-2853.
3. K. Divaani-Aazar and A. Mafi, Associated Primes of Local Cohomology Modules of Weakly Laskerian Modules, Comm. in Algebra 34(2006), 1532-4125.
4. E. Enochs, Flat covers and flat cotorsion modules, Proc. Amer. math. Soc. 92 (1984) 179-184.
5. P. Hung Quy, On the finiteness of associated primes of local cohomology modules, Proc. Amer. Math. Soc. in press.
6. T. Marley and J. C. Vassilev, Cofiniteness and associated primes of local cohomology modules, J. Algebra 256(1) (2002), 180-193.
7. H. Matsumura, Commutative Ring Theory, in: Cambridge Stud. Adv. Math., Vol. 8, Cambridge Univ. Press, Cambridge, 1986.

8. T. Zink, Endlichkeitsbedingungen für Moduln über einem Notherschen Ring, Math. Nachr. 164 (1974) 239-252.
9. H. Zöschinger, Minimax modules, J. Algebra. 102 (1986) 1-32.
10. H. Zöschinger, Über die Maximalbedingung für radikalvolle Untermoduln, Hokkaido Math. J. 17 (1988) 101-116.

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